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Four-Dimensional, Ricci-Flat Manifolds which Admit a Metric

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Abstract. This paper discusses the relationships between the metric, the connection and the curvature tensor of 4-dimensional, Ricci flat manifolds which admit a metric. It is shown that these metric and curvature objects are essentially equivalent conditions for such manifolds if one excludes certain very special cases and which occur when the signature is indefinite. In a similar vein, some relevant remarks are made regarding the Weyl conformal tensor.

1. Introduction

In this paper, (M, g) denotes a 4-dimensional, smooth, connected manifold with metric g of signature $(+,+,+,+)$, $(+,+,+,-)$ or $(+,+,-,-)$, referred to, respectively, as positive definite, Lorentz or neutral. The tangent space to M at $m \in M$ is denoted by T_mM. The Levi-Civita connection arising from g is denoted by ∇, the corresponding type $(1,3)$ curvature tensor by *Riem* with components $R^a{}_{bcd}$, the Ricci tensor by *Ricc*, with components $R_{ab} \equiv R^c{}_{acb}$, and the Ricci scalar by $R \equiv g^{ab}R_{ab}$. Since ∇ is assumed to be a metric connection, one also has the type $(0,4)$ curvature tensor with components $R_{abcd} \equiv g_{ae}R^e{}_{bcd}$. The pair (M, g) is called *non-flat* if *Riem* does not vanish over any non-empty open subset of M and *not flat* is the negation of *flat*. Thus in the non-flat case the subset of M on which *Riem* does not vanish is open and dense in M. In the analytic case, the terms non-flat and not-flat are equivalent but analyticity will not be assumed here. The type $(1,3)$ Weyl *conformal* tensor is labelled C with components $C^a{}_{bcd}$

If $p, q \in T_mM$, $g(m)(p,q)$ is denoted by $p \cdot q$ and a non-zero $p \in T_mM$ is called *spacelike*, *timelike* or *null* if $p \cdot p > 0$, $p \cdot p < 0$ or $p \cdot p = 0$, respectively. If g is positive definite one may choose a basis x, y, z, w of mutually orthogonal, unit vectors in T_mM whereas if g has Lorentz signature one may choose a basis x, y, z, t for T_mM which is mutually orthogonal and $x \cdot x = y \cdot y = z \cdot z = -t \cdot t = 1$ and an associated null basis l, n, x, y where $\sqrt{2}l = z + t$ and $\sqrt{2}n = z - t$ so that the null vectors l and n satisfy $l.n = 1$. If g has neutral signature $(+,+,-,-)$ one may choose a basis of mutually orthogonal vectors x, y, s, t for T_mM such that $x.x = y.y = -s.s = -t.t = 1$ and an associated null basis l, n, L, N where $\sqrt{2}l = x + t$, $\sqrt{2}n = x - t$, $\sqrt{2}L = y + s$ and $\sqrt{2}N = y - s$ so that l, n, L, N are each null and $l.n = L.N = 1$.

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2. Bivectors

Let $\Lambda_m M$ denote the 6-dimensional space of 2-forms (*bivectors*) at m, disregarding their tensor type (position of tensor indices) due to the existence of a metric. Let [ ] denote the product on $\Lambda_m M$ given by matrix commutation and $<>$ the inner product on $\Lambda_m M$ given by $< F, G >\equiv F_{ab}G^{ab}$. Thus $\Lambda_m M$ is a Lie algebra under [ ]. If $0 \neq F \in \Lambda_m M$ with components F^{ab}, F has even (matrix) rank and hence has rank 2 or 4. If F has rank 2, it is called *simple* and otherwise, *non-simple*. If F is simple it may be written as $F = p \wedge q$ for $p, q \in T_m M$ (in components, $F^{ab} = p^a q^b - q^a p^b$) and the *span* of p and q, denoted by $< p, q >$, is uniquely determined by p and q and called the *blade* of F. Sometimes it is convenient to denote either F or its blade by $p \wedge q$. A simple bivector is called *spacelike*, *timelike*, *null* or *totally null* if its blade is a spacelike (that is, spanned by two orthogonal non-null vectors of the same sign), a timelike (spanned by two non-orthogonal null vectors), a null (spanned by two orthogonal vectors one of which is null and the other non-null) or a *totally null* (spanned by two orthogonal null vectors) 2−dimensional subspace (2−space) of $T_m M$. Only the first of these can occur for positive definite, the first three for Lorentz (with the vectors spanning the spacelike case being both spacelike and, in the null case, the non-null vector is necessarily spacelike), and all four in the case of neutral, signature (with either sign being possible in the spacelike case and, in the null case, the non-null vector may also be of either sign). If the signature is positive definite, any simple bivector F may be written in some orthonormal basis as a multiple of $x \wedge y$ and any non-simple one as $\alpha(x \wedge y) + \beta(z \wedge w)$ for $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq 0 \neq \beta$. For signature $(+, +, +, -)$ appropriate orthogonal or null bases may be chosen so that a spacelike bivector may be written as a multiple of $x \wedge y$, a timelike one as some multiple of $z \wedge t$ (or of $l \wedge n$), a null one as a multiple of $l \wedge x$ and a non-simple one as $F = \alpha(x \wedge y) + \beta(l \wedge n)$ for $\alpha \neq 0 \neq \beta$. In this latter non-simple case the 2-spaces $x \wedge y$ and $l \wedge n$ are uniquely determined by F [2]. Also for this signature, a simple bivector is null if and only if $< F, F >= 0$. For signature $(+, +, -, -)$ bases may be chosen so that a null bivector F may be written as a multiple of $l \wedge y$ or of $l \wedge s$ and a *totally null* one as $F = l \wedge L$ (and then a simple F is null *or* totally null if and only if $< F, F >= 0$). For this signature non-simple bivectors are a little more complicated [1, 5, 11] but this is not important in what is to follow.

Let $*$ denote the usual Hodge "duality" operator as the linear map on $\Lambda_m M$ given by $F \to \overset{*}{F}$. For positive definite and neutral signatures, $\overset{**}{F} = F$ whereas for Lorentz signature, $\overset{**}{F} = -F$. In all cases, F is simple if and only if $\overset{*}{F}$ is. Define the subspaces $\overset{+}{S}_m = \{F : \overset{*}{F} = F\}$, $\overset{-}{S}_m = \{F : \overset{*}{F} = -F\}$ and $\widetilde{S_m} \equiv \overset{+}{S}_m \cup \overset{-}{S}_m$ of $\Lambda_m M$. If one has Lorentz signature, $\overset{+}{S}_m$, $\overset{-}{S}_m$ and $\widetilde{S_m}$ are trivial subspaces and so F and $\overset{*}{F}$ are independent members of $\Lambda_m M$ for any $F \in \Lambda_m M$. For the other two signatures each bivector F may be written uniquely as $F = \overset{+}{F} + \overset{-}{F}$ where $\overset{+}{F} \in \overset{+}{S}_m$ and $\overset{-}{F} \in \overset{-}{S}_m$ and so $\Lambda_m M = \overset{+}{S}_m \oplus \overset{-}{S}_m$. Further, for $\overset{+}{F} \in \overset{+}{S}_m$ and $\overset{-}{F} \in \overset{-}{S}_m$, $[\overset{+}{F}, \overset{-}{F}] = 0$ and $< \overset{+}{F}, \overset{-}{F} >= 0$. One thus sees that the Lie algebra $\Lambda_m M$ with product [ ] is the sum of the Lie algebras $\overset{+}{S}_m$ and $\overset{-}{S}_m$. For positive definite signature these latter two Lie algebras are isomorphic to $o(3)$ and for neutral signature, to $o(1, 2)$.

For positive definite signature, any member F of $\widetilde{S_m}$ is non-simple and may be written as $F = \alpha(x \wedge y) + \beta(z \wedge w)$ with $\alpha = \pm\beta$ whereas any *non-simple* member of $\Lambda_m M \setminus \widetilde{S_m}$ can be written in this form but with $\alpha \neq \pm\beta$. In the latter case F uniquely determines the two 2−spaces $x \wedge y$ and $z \wedge w$ but this is not true if $F \in \widetilde{S_m}$ [11]. For the basis choice $(x \wedge y)^* = z \wedge w$, $\overset{+}{S}_m$ is spanned by $(x \wedge y) + (z \wedge w)$, $(x \wedge z) + (w \wedge y)$ and $(x \wedge w) + (y \wedge z)$ and a similar basis is easily constructed for $\overset{-}{S}_m$. For neutral signature and $F \in \widetilde{S_m}$ F is either totally null or non-simple and with the basis choice $(x \wedge y)^* = s \wedge t$, a spanning set for $\overset{+}{S}_m$ is $(l \wedge n) - (L \wedge N)$, $l \wedge N$ and $n \wedge L$ with a similar one for $\overset{-}{S}_m$. The first two members of this spanning set span a 2−dimensional subalgebra of $\overset{+}{S}_m$ denoted by $\overset{+}{B}_m$ (and similarly in $\overset{-}{S}_m$). Of course, there are no 2−dimensional subalgebras of $\overset{+}{S}_m$ or $\overset{-}{S}_m$ in the positive definite case. Further details on such matters may be found in [1, 5, 11] whilst the Lorentz case is discussed in [4].

3. The Ricci flat condition

If one imposes the *Ricci-flat* condition on (M, g), so that $Ricc \equiv 0$ on M, then $Riem = C$ on M and so the Hodge dual condition shows (see, e.g. [6]) that the left and right duals ${}^*Riem$ and $Riem^*$ of $Riem$ are equal which, in components, gives ${}^*R_{abcd} = R^*_{abcd}$. Now define the curvature map f from tensor type $(2,0)$ members of $\Lambda_m M$ to tensor type $(1,1)$ members of it as the (linear) map given in components by; $f : F^{ab} \to R^a{}_{bcd}F^{cd}$. Because of the (Ricci-flat) dual condition on $Riem$, this map is easily checked to satisfy the condition that $F \in \overset{+}{S}_m$ (respectively $F \in \overset{-}{S}_m$) implies that $f(F) \in \overset{+}{S}_m$ (respectively $f(F) \in \overset{-}{S}_m$) and so $\overset{+}{S}_m$ and $\overset{-}{S}_m$ are *invariant subspaces* of f. (In fact, this invariance property is equivalent to (M, g) being an Einstein space [12].) One advantage of this is that one may choose a basis for the range space, $\mathrm{rg}f$, of f which is composed of members of $\widetilde{S_m}$. Thus if F_1, F_2, and F_3 constitute a basis for $\overset{+}{S}_m$ and G_1, G_2, and G_3 a basis for $\overset{-}{S}_m$ then for signatures $(+,+,+,+)$ and $(+,+,-,-)$ one may express the curvature, in an obvious notation, as

$$Riem(m) = \sum_{i,j=1}^{3} (\alpha_{ij} F^i F^j + \beta_{ij} G^i G^j) \tag{1}$$

for symmetric arrays of real numbers α_{ij} and β_{ij}. From the Ricci-flat condition it also follows from the dual condition on $Riem$ that, for *Lorentz* signature, $\mathrm{rg}f$ has even dimension (see [4], chapter 9). Such a result applies also to the analogous map constructed from the Weyl tensor on any 4−dimensional manifold (not necessarily Ricci-flat) because of the trace free condition $C^c{}_{acb} = 0$ on the Weyl tensor and is important in the Petrov classification of gravitational fields in general relativity theory [3] (see also [4, 7]).

4. Some useful lemmas

In this section a few technical results will be gathered together for later use. More details and proofs may be found in [1, 4, 5, 11].

Let $F \in \Lambda_m M$ and consider the vector space of solutions for a second order symmetric tensor h at m of the equation

$$h_{ac}F^c{}_b + h_{bc}F^c{}_a = 0. \tag{2}$$

(Clearly $h = g$ is always a solution.) Then

Lemma 4.1. *For any signature of g, if F is simple, h satisfies (2) if and only if the blade of F is an eigenspace of h with respect to g, that is, $h_{ab}p^b = \lambda g_{ab}p^b$ for each p in the blade of F and for $\lambda \in \mathbb{R}$.*

Lemma 4.2. *If g has Lorentz signature and F is non-simple then h satisfies (2) if and only if the two, uniquely determined, orthogonal 2-spaces $l \wedge n$ and $x \wedge y$ given in section* 2 *are each eigenspaces of h with respect to g.*

Lemma 4.3. *Suppose g has positive definite signature and F is non-simple and of the form $F = \alpha(x \wedge y) + \beta(z \wedge w)$ with $\alpha \neq 0 \neq \beta$. Then if $\alpha \neq \pm\beta$, $F \notin \widetilde{S_m}$ and h satisfies (2) if and only if the uniquely determined, orthogonal 2-spaces $x \wedge y$ and $z \wedge w$ (section* 2*), are eigenspaces of h with respect to g. If $\alpha = \pm\beta$ then $F \in \widetilde{S_m}$ and* there exist *two orthogonal 2-spaces which are eigenspaces of h with respect to g. If $F_1, F_2 \in \overset{+}{S}_m$ determine the same pair of eigenspaces as in the previous sentence (for fixed h), F_1 and F_2 are proportional, and similarly for $\overset{-}{S}_m$*

Lemma 4.4. *Suppose g has neutral signature. If F is non-simple and of the form $F = \lambda(l \wedge n) + \nu(L \wedge N)$ with λ and ν non-zero then, if $\lambda \neq \pm\nu$, $F \notin \widetilde{S_m}$ and h satisfies (2) if and only if the two 2-spaces $l \wedge n$ and $L \wedge N$ are each eigenspaces of h with respect to g whereas if $\lambda = -\nu$, ($F \in \overset{+}{S}_m$), h satisfies (2) if and only if the two 2-spaces $l \wedge N$ and $n \wedge L$ are each* invariant *2-spaces of h with respect to g (with a similar result holding if $\lambda = \nu$ and $F \in \overset{-}{S}_m$).*

5. Holonomy theory and holonomy algebras

In what is to follow, much use will be made of holonomy theory and full details may be found in [4, 13]. The *holonomy group* of (M, g) is a Lie group denoted by Φ with Lie algebra ϕ and there is a bivector representation of this holonomy algebra. Associated with these constructions is the *infinitesimal holonomy algebra* ϕ'_m at $m \in M$ which is a subalgebra of ϕ containing $\mathrm{rg}f(m)$ as a subspace. Thus *Riem* may always be expressed, at any point, as (symmetrised) products of bivectors drawn from the (bivector representation of the) appropriate holonomy algebra. An important link between ϕ, ϕ_m and $\mathrm{rg}f$ is provided by the Ambrose-Singer theorem [14] (see also, [12, 13]). This theorem states that if one chooses a fixed $m \in M$ and then, for any $m' \in M$, parallel transports $\mathrm{rg}f$ along a smooth curve c from m' to m (since M is connected and hence path connected) and does this for all $m' \in M$ and all such curves c, one accumulates, at m, a spanning set of bivectors for ϕ. The holonomy algebra of (M, g) is a subalgebra of the appropriate orthogonal algebra of g, that is, of $o(4)$, $o(1, 3)$ and $o(2, 2)$, for signatures $(+, +, +, +)$, $(+, +, +, -)$ and $(+, +, -, -)$, respectively. In the positive definite case the subalgebras are, up to isomorphism, (cf [5]) $\{x \wedge y\}$ (labelled S_1), $\{x \wedge y, z \wedge w\}$ (S_2), $\{x \wedge y, x \wedge z, y \wedge z\}$ (S_3), $\{\overset{+}{S}\}$ $(\overset{+}{S}_3)$, $\{\overset{+}{S}, G\}$ with $G \in \overset{-}{S}_m$ $(\overset{+}{S}_4)$ and $o(4)$ (S_6), where $\overset{+}{S}$ denotes $o(3)$ and $\{\}$ denotes a spanning set. For the Lorentz case the holonomy subalgebras have been tabulated in [4, 8, 9]. For neutral signature they are given in table 1 ([1], see also [10]) in which $\alpha, \beta \in \mathbb{R}$ and $\alpha \neq 0 \neq \beta$ hold in case $2(j)$, $\alpha \neq \pm\beta$ holds in cases $2(h)$ and $3(d)$, $\overset{+}{S}$ is the algebra $o(1, 2)$ and $\overset{+}{B}$ is the 2-dimensional algebra given earlier in the paper. For each signature the trivial holonomy algebra ($\Rightarrow$ *Riem* $= 0$ on M) is omitted as is the case when the holonomy algebra is 1-dimensional and spanned by a non-simple bivector F since this cannot occur for a *metric* connection because of the algebraic identity $R_{a[bcd]} = 0$ which would force the spanning bivector to satisfy $F_{a[b}F_{cd]} = 0$ and hence to be simple. Here, square brackets denote the usual skew-symmetrisation of the enclosed indices. Table 1 consists of 23 of the 32 cases listed in the table in [10] being those which apply (or, in two cases, may apply) to metric connections.

It is remarked that if $k \in T_mM$ is an eigenvector of each member of the spanning set of bivectors, with respect to g, for a certain holonomy algebra, there exists a local *recurrent* vector field X on some coordinate neighbourhood U of m and such that $X(m) = k$ (that is, X satisfies $X_{a;b} = X_aP_b$ on U for some smooth 1-form field P on U, where a semi-colon denotes a covariant derivative with respect to the Levi-Civita connection from g). Further, if such a k exists and if each of the above mentioned eigenvalues is zero, then X may be chosen to be covariantly constant. In fact, any non-null recurrent vector field may be rescaled so that it is covariantly constant, this result following from the fact that such a rescaling is equivalent to the 1-form P being locally a gradient and also equivalent to the condition $R^a{}_{bcd}X^d = 0$, on U.

6. Main results

Suppose now that M is a $4-$dimensional smooth manifold with metric g of arbitrary signature satisfying the Ricci-flat condition. If the holonomy algebra of (M, g) is $1-$dimensional and spanned by the bivector F then since it is not flat, there exists $m \in M$ such that $Riem(m) \neq 0$ and so, at m, $R_{abcd} = \alpha F_{ab}F_{cd}$ for $0 \neq \alpha \in \mathbb{R}$. As above, the algebraic identity $R_{a[bcd]} = 0$ shows that F is simple and an easy substitution of the possibilities for a simple bivector into this expression for the curvature and imposition of the Ricci-flat condition show that the only possibility is that F is totally null and hence that g is of neutral signature. Thus the holonomy algebra is of type $1(d)$ in table 1. The following theorems can now be proved (more details being available in [12]).

Theorem 6.1. *Suppose dimM* $= 4$ *and g is of positive definite signature. Then*

(i) If the holonomy type of (M, g) is $\overset{+}{S}_3$, $\overset{+}{S}_4$ or S_6 the Levi-Civita connection ∇ associated with g determines g up to a constant conformal factor.

(ii) If (M, g) is Ricci-flat and not flat, ∇ determines g up to a constant conformal factor.

(iii) If (M, g) is Ricci-flat and non-flat, Riem determines g up to a constant conformal factor and hence determines ∇ uniquely.

Table 1: Holonomy algebras for $(+,+,-,-)$

Type	Dimension	Basis
1(*a*)	1	$l \wedge n$
1(*b*)	1	$x \wedge y$
1(*c*)	1	$l \wedge y$ or $l \wedge s$
1(*d*)	1	$l \wedge L$
2(*a*)	2	$l \wedge n - L \wedge N, l \wedge N (= \overset{+}{B})$
2(*b*)	2	$l \wedge n, L \wedge N$
2(*c*)	2	$l \wedge n - L \wedge N, x \wedge y - s \wedge t$
2(*d*)	2	$l \wedge n - L \wedge N, l \wedge L$
2(*e*)	2	$x \wedge y, s \wedge t$
2(*f*)	2	$x \wedge y + s \wedge t, l \wedge L$
2(*g*)	2	$l \wedge N, l \wedge L$
2(*h*)	2	$l \wedge N, \alpha(l \wedge n) + \beta(L \wedge N)$
2(*j*)	2	$l \wedge N, \alpha(l \wedge n - L \wedge N) + \beta(l \wedge L)$
2(*k*)	2	$l \wedge y, l \wedge n$ or $l \wedge s, l \wedge n$
3(*a*)	3	$l \wedge n, l \wedge N, L \wedge N$
3(*b*)	3	$l \wedge n - L \wedge N, l \wedge N, l \wedge L$
3(*c*)	3	$x \wedge y, x \wedge t, y \wedge t$ or $x \wedge s, x \wedge t, s \wedge t$
3(*d*)	3	$l \wedge N, l \wedge L, \alpha(l \wedge n) + \beta(L \wedge N)$
4(*a*)	4	$\overset{+}{S}, l \wedge n + L \wedge N$
4(*b*)	4	$\overset{+}{S}, x \wedge y - s \wedge t$
4(*c*)	4	$\overset{+}{B}, \overset{-}{B} = < l \wedge L, l \wedge N, l \wedge n, L \wedge N >$
5	5	$\overset{+}{S}, \overset{-}{B}$
6	6	$o(2,2)$

Proof. (i) Suppose that g' is another metric on M whose Levi-Civita connection ∇' is the same as that of g, $\nabla = \nabla'$. Then their holonomy algebras are the same and each member F of this latter algebra satisfies (2) with $h = g$ and $h = g'$. If this holonomy algebra is $\overset{+}{S}_3$, lemma 4.3 shows that two (in fact three) distinct pairs of orthogonal 2−spaces are eigenspaces for g' with respect g and hence that T_mM is an eigenspace of g' with respect to g. Thus g' and g are conformally related. The conditions $\nabla g = 0$ and $\nabla' g = 0$ and the connectedness of M then show that the conformal factor is constant. If the holonomy algebra is $\overset{+}{S}_4$ or S_6 the same follows since each contains $\overset{+}{S}_3$ as a subalgebra.

(ii) It is sufficient to show that the holonomy algebras S_1, S_2 and S_3 cannot occur in the Ricci-flat case. To see this one notes that a remark at the beginning of this section shows that the 1−dimensional case S_1 cannot occur. If the holonomy algebra is S_2 there exists $m \in M$ where $Riem \neq 0$ and so the range of the curvature map f at m is spanned by bivectors $F = x \wedge y$ and $G = z \wedge w$. Then one gets

$$R_{abcd} = \alpha F_{ab}F_{cd} + \beta G_{ab}G_{cd} + \gamma(F_{ab}G_{cd} + G_{ab}F_{cd}) \tag{3}$$

for real numbers α, β and γ. The Ricci-flat condition $R^c{}_{acb} = 0$ shows that $\alpha = \beta = 0$ and a contraction of the identity $R_{a[bcd]} = 0$ with $z^a w^b$ gives the contradiction $\gamma = 0$. For the holonomy algebra S_3, the argument is similar.

(iii) If g and g' are metrics on M such that, in an obvious notation, $Riem' = Riem$, then the Ricci-flat condition on g imposes the Ricci-flat condition on g' and so the Weyl tensors of g and g' are equal (say to C) and the non-flat condition implies that C does not vanish over any non-empty open subset of M. It thus follows [11] that g and g' are conformally related, $g' = \psi g$, for some smooth real-valued function ψ on M. The Bianchi identities for (M, g) and (M, g') are

$$R^a{}_{bcd;a} = 0 \qquad R^a{}_{bcd|a} = 0 \tag{4}$$

where a stroke denotes a covariant derivative with respect to g'. Since $g' = \psi g$, the Christoffel symbols in this coordinate system associated with ∇ and ∇' are related, in an obvious notation, in the following way

$$P^a_{bc} \equiv \Gamma'^a_{bc} - \Gamma^a_{bc} = \frac{1}{2}\psi^{-1}(\psi_{,c}\delta^a_b + \psi_{,b}\delta^a_c - \psi^a g_{bc}) \tag{5}$$

where a comma denotes a partial derivative and $\psi^a = g^{ab}\psi_{,b}$. A subtraction of the two expressions in (4) yields

$$R^e{}_{bcd}P^a_{ea} - R^a{}_{ecd}P^e_{ba} - R^a{}_{bed}P^e_{ca} - R^a{}_{bce}P^e_{da} = 0 \tag{6}$$

A substitution of (5) into (6) using the algebraic Bianchi identity $R^a{}_{[bcd]} = 0$ gives, after some calculation, $R^a{}_{bcd}\psi^d = 0$. From the definition of the curvature map f it follows from this last relation that for m in the open dense subset of M where $Riem$ does not vanish, if $\psi_{,a}(m) \neq 0$ the range of the curvature map rgf at m contains only simple bivectors and this is a contradiction to the fact that, for this signature, the range of f admits a basis consisting of members of $\widetilde{S_m}$ each of which is non-simple. It follows that $\psi_{,a}$ vanishes over this open dense subset and hence on M. Since M is connected, ψ is constant on M. It is noted that the Ricci-flat condition is not used in part (i) of this theorem. □

Before the Lorentz case is considered, a few words should be said about the Petrov classification of gravitational fields in general relativity theory [3] (see also [4, 7]). The details of this classification are not needed here except to say that it is a pointwise algebraic classification of the Weyl tensor C into a number of exhaustive and mutually exclusive (Petrov) types and that one of these types, labelled **N**, is equivalent to the condition that, at the point $m \in M$ in question, C is not zero and satisfies $C^a{}_{bcd}k^d = 0$ for some $0 \neq k \in T_mM$ (and the direction spanned by k is necessarily unique and null). The Petrov type is labelled

O at m if $C(m) = 0$. If the Ricci-flat condition is assumed, $Riem = C$ and the Petrov classification applies to $Riem$ (This is the important *vacuum* case in Einstein's general theory of relativity.) It can then be checked from the actual Petrov algebraic types themselves (see e.g. [4]) that the rank of the curvature map f, that is, $\dim(\mathrm{rg}f)$, exceeds 2 if and only if the Petrov type is not **N** or **O**. Quite generally, *for any signature of g* but still with the Ricci-flat assumption, let V denote the subset of M consisting of those points $m \in M$ at which the equation $R^a{}_{bcd}k^d = 0$ has a nontrivial solution for $0 \neq k \in T_mM$ (so that V includes those points at which $Riem$ vanishes). In the positive definite case, V consists of precisely those points at which $Riem$ vanishes otherwise any member $F \in \mathrm{rg}f$ would satisfy $F_{ab}k^b = 0$ and hence would be simple, contradicting the fact that a basis for $\mathrm{rg}f$ can be chosen from members of $\widetilde{S_m}$, each of which is non-simple. In the Lorentz case, V consists of precisely those points where the Petrov type is **O** or **N**. If g has neutral signature, again $\mathrm{rg}f$ consists entirely of simple bivectors and again a basis for $\mathrm{rg}f$ can be chosen from members of $\widetilde{S_m}$. For this signature, however, there exist simple members of $\widetilde{S_m}$ and they are necessarily totally null (section 2). If $\mathrm{rg}f$ contains two or more members of $\overset{+}{S_m}$ (or of $\overset{-}{S_m}$) it is easily checked that non-simple members will result in $\mathrm{rg}f$. Thus, if $m \in V$ either $Riem(m) = 0$, *or* $\mathrm{rg}f$ is spanned by a totally null bivector which, in some null basis, may be chosen, up to isomorphism, as $l \wedge N$ (and in which case l and N are the (only independent) solutions for k above) *or* $\mathrm{rg}f$ is spanned by two totally null bivectors one from each of $\overset{+}{S_m}$ and $\overset{-}{S_m}$. In this last case it is straightforward to show that a null basis exists in which they are $l \wedge N$ and $l \wedge L$ and so l is the (only independent) solution for k [1]. It is not hard to check that, in the Ricci-flat case and for any signature, the set V is precisely the set of all points of M at which each member F in $\mathrm{rg}f$ satisfies $< F, F >= 0$ and so V is closed in M.

Theorem 6.2. *[4] Suppose $\dim M = 4$ and g is of Lorentz signature.*

(i) If the holonomy type of (M, g) is R_9, R_{12}, R_{14} or R_{15} the Levi-Civita connection ∇ of g uniquely determines g up to a constant conformal factor.

(ii) If (M, g) is Ricci-flat but not flat, the holonomy type is R_8, R_{14} or R_{15}.

(iii) If (M, g) is Ricci-flat and is such that there exists $m \in M$ at which the Petrov type is not **N** *or* **O**, *(that is, $M \neq V$) ∇ determines g up to a constant conformal factor.*

(iv) If (M, g) is Ricci-flat and the subset V has empty interior in (the manifold topology of) M, Riem determines g up to a constant conformal factor (and hence determines ∇ uniquely).

It is noted that the Ricci-flat conditon is not used in part (i) of this theorem.

Theorem 6.3. *Suppose $\dim M = 4$ and g is of neutral signature.*

(i) If the holonomy type of (M, g) is 2(a), 2(h) (with $\alpha \neq 0 \neq \beta$), 3(a), 3(b), 3(d) (with $\alpha \neq 0$), 4(a), 4(b), 4(c), 5 or 6, the Levi-Civita connection associated with g determines g up to a constant conformal factor.

(ii) If (M, g) is Ricci-flat but not flat, its holonomy type cannot be any of the types 1(a), 1(b), 1(c), 2(b), 2(c), 2(d), 2(e), 2(f), 2(h), 2(j), 2(k), 3(c) or 3(d) ($\alpha = 0$), (at least).

(iii) If (M, g) is Ricci-flat but not flat and if its holonomy type is not 1(d) or 2(g), its associated Levi-Civita connection uniquely determines g up to a constant conformal factor.

(iv) If (M, g) is Ricci-flat and suppose that the set V has empty interior in M. Then Riem determines g up to a constant conformal factor (and hence determines ∇ uniquely).

Proof. (i) The proof of this part consists of checking the various holonomy types in table 3 and proceeding in much the same way as in theorem 6.1(i). For example, if the holonomy type is 2(a) in table 3 and g and g' are compatible metrics for ∇, the bivectors in the representation for this holonomy are, in some suitable null basis, $F = l \wedge n - L \wedge N$ and $G = l \wedge N$. Then (2) holds for F and G for each of g and g' and an appeal to lemma 4.1 for G and lemma 4.4 for F shows that $l \wedge N$ is an eigenspace of g' with respect g and that $l \wedge N$ and $n \wedge L$ are invariant 2−spaces of g' with respect g. This reveals that T_mM is an eigenspace of g' with respect to g and so g is conformally related to g' at each $m \in M$. Again, since each is compatible with ∇ and M is connected, the result is achieved. The other cases are dealt with in a (more or less) similar manner, and noting that in each of the 4−, 5− and 6−dimensional cases the holonomy algebra contains $\overset{+}{B}$ (that is, 2(a)) as a subalgebra and the result follows from the 2(a) case.

(ii) The general idea here is to write down an expression for R_{abcd} at $m \in M$ in terms of the basis members of the holonomy algebra, remembering (1), and using the conditions $Ricc = 0$, $R_{a[bcd]} = 0$, the facts that $\overset{+}{S}_m$ and $\overset{-}{S}_m$ are invariant under f and, occasionally, that recurrent vector fields exist and the Ambrose-Singer theorem. It is unfortunately a little lengthy and more details will be published elsewhere [12]. Thus subalgebras 1(a), 1(b) and 1(c) are easily ruled out. For type 2(b) it follows that, at each $m \in M$, (3) holds with $F = l \wedge n$ and $G = L \wedge N$. The Ricci-flat condition then gives $\alpha = \beta = 0$ and $R_{a[bcd]} = 0$, after a contraction with l^a, gives either $\gamma = 0$ or $l_{[a}G_{bc]} = 0$. The first contradicts the fact that (M, g) is not flat and the second gives the contradiction that G is simple with l in its blade. So this type is not possible.

(iii) This now follows from the previous two parts.

(iv) If g and g' have the same tensor $Riem$, g' is also Ricci-flat and g and g' have the same tensor C. Then it follows from the conditions of the theorem [15] that g and g' are conformally related and the result now follows from a similar calculation to that performed in the proof of theorem 6.1(iii).

Again, the Ricci-flat condition is not used in part (i) of this theorem. □

In conclusion one can state the following theorem which covers all signatures in the 4−dimensional case.

Theorem 6.4. *Suppose dimM = 4 with g of arbitrary signature and with (M, g) Ricci-flat. Suppose also that the closed set V has empty interior. Then ∇ determines g up to a constant conformal factor and Riem determines g up to a constant conformal factor and hence determines ∇ uniquely.*

Thus under the stated conditions, the metric, its Levi-Civita connection and the latter's curvature tensor are "equivalent". It is also interesting to remark that, from a slight reworking of this proof, it follows that if the (closed) set V', defined as the subset of all points of M at which the equation $C^a{}_{bcd}k^d = 0$ has non-trivial solutions for $k \in T_mM$, has empty interior, the Weyl conformal tensor C uniquely determines the *conformal class* of g [4, 11, 15]. Here one uses the (identity) tracefree condition on C as a replacement for the Ricci-flat condition and the obvious analogy for C of the curvature map f. It is remarked that theorem 6.4 fails if the Ricci-flat condition is replaced by the Einstein space condition [12].

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